

Characterization of n -associative, monotone, idempotent functions on an interval which have neutral elements

Gergely Kiss ^{*1} and Gábor Somlai ^{†2}

¹University of Luxembourg, Faculty of Science, Mathematical Research Unit

²Eötvös Loránd University, Faculty of Science, Institute of Mathematics

Abstract

We investigate monotone idempotent n -ary semigroups. One of the main result of this article is the generalisation of Czogala-Drewniak Theorem, which describes the idempotent monotone associative functions having neutral element. Furthermore we present the full characterisation of idempotent, monotone, n -associative functions on an interval which have neutral elements. Our description provides that monotone, idempotent, n -ary semigroups are quasitrivial.

1 Introduction

A function $F : X^n \rightarrow X$ is called n -associative if for every $x_1, \dots, x_{2n-1} \in X$ and for every $1 \leq i \leq n-1$ we have

$$\begin{aligned} F(F(x_1, \dots, x_n), x_{n+1}, \dots, x_{2n-1}) = \\ F(x_1, \dots, x_i, F(x_{i+1}, \dots, x_{i+n}), x_{i+n+1}, \dots, x_{2n-1}). \end{aligned} \quad (1)$$

Throughout this paper we assume that the underlying sets of the observed algebraic structures are totally ordered. In our main results we investigate n -ary semigroups on an arbitrary nonempty subinterval of the real numbers.

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A set X endowed with an n -associative function $F : X^n \rightarrow X$ is called an n -ary semigroup and is denoted by (X, F_n) . The main purpose of this paper is to describe a class of n -ary semigroups. An n -ary semigroup is called *idempotent* if $F(a, \dots, a) = a$ for all $a \in X$. Another important property is the monotonicity. An n -associative function is called *monotone* if it is monotone in each of its variables. Further we say that an n -associative function has *neutral element* denoted by $e \in X$ if for every $x \in X$ and $1 \leq i \leq n$ we have $F(e, \dots, e, x, e, \dots, e) = x$, where x is substituted into the i 'th coordinate.

An important construction of n -ary semigroups is the following. Let (X, F_2) be a binary semigroup. Let $F_n := \underbrace{F_2 \circ F_2 \circ \dots \circ F_2}_{n-1}$. Using the associativity of

F_2 one might choose the way of substituting variables x_1, \dots, x_n into the right hand side of the equation and obtain the same element of X . This shows that the previous expression is well-defined. Therefore we obtain an n -associative function $F_n : X^n \rightarrow X$ and an n -ary semigroup (X, F_n) . In this case we say that (X, F_n) is *derived* from the binary semigroup (X, F_2) .

Generally we simply say that F_n is derived from F_2 . We also use notation (X, F_n) is a *totally ordered n -ary semigroup* for emphasising that X is totally ordered.

It is easy to show (see Lemma 3.1) that if F_n is derived from F_2 and F_2 is either monotone or idempotent or has a neutral element then F_n has the same property.

An n -ary semigroup (X, F_n) is called an n -ary group if for every $n - 1$ elements $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ in X and for every $a \in X$ there exists a unique $b \in X$ with $F_n(x_1, \dots, x_{i-1}, b, x_{i+1}, \dots, x_n) = a$. It is easy to see from the definition that ordinary groups are exactly the 2-ary groups. Clearly, a function F_n derived from a semigroup F_2 is n -associative but not every n -ary semigroup can be obtained in this way. Dudek and Mukhin [3] proved that an n -ary semigroup X is derived from a binary one if and only if X contains or one can adjoin a neutral element. As a special case of this theorem they obtained that X is an n -ary group which is derived from a group if and only if it contains a neutral element.

This theorem allows us to construct n -ary groups which are not derived from binary groups if n is odd. Indeed, let $G_n(x_1, \dots, x_{2n-1}) = \sum_{i=1}^{2n-1} (-1)^i x_i$. It is easy to verify that G_n is n -associative and we obtain an n -ary group. Moreover G_n is clearly monotone. It is also easy to check that there is no neutral element for G_n .

Finally, we say that an n -ary semigroup (X, F_n) is *quasitrivial* if for every $x_1, \dots, x_n \in X$ we have $F_n(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}$. Such an n -variable function F_n is called a *choice function*. Ackerman (see [1]) investigated quasitrivial semigroups and also gave a characterization of them.

Our paper is organised as follows. In Section 2 we collect the main results proved in the paper. In Section 3 we establish connections between n -ary semigroups and binary semigroups and we prove Theorem 2.3. Section 4 is devoted to the proof of Theorem 2.4 and Theorem 2.5. Section 5 contains our concluding

remarks.

2 Main results

Let $I \subset \mathbb{R}$ be a not necessarily bounded, nonempty interval and $g : I \rightarrow I$ be a decreasing function. If x is a discontinuity point of g , then $g(x - 0)$ and $g(x + 0)$ denote limit of g from the left and from the right, respectively. We denote by Γ_g the *completed graph* of g , which means that if x is a discontinuity point of g , then we put a vertical line segment between the points $(x, g(x - 0))$ and $(x, g(x + 0))$. Formally,

$$\Gamma_g = \{(x, y) \in I^2 : g(x + 0) \leq y \leq g(x - 0)\}.$$

We call Γ_g (*id-*)*symmetric* if Γ_g is symmetric to the line $x = y$.

The following theorem gives a description of idempotent, monotone, (2-ary) semigroups with neutral elements. These semigroups were first investigated by Czogala and Drewniak [2], where the authors only dealt with closed subintervals of \mathbb{R} but the statement holds for any non-empty interval. On the other hand, instead of monotonicity it was assumed that the binary function is monotone increasing. However, Lemma 3.7 shows that monotonicity implies monotone increasingness in this case.

Theorem 2.1. *Let I be an arbitrary nonempty real interval. If a function $F_2 : I^2 \rightarrow I$ is associative, idempotent, monotone which has a neutral element $e \in I$, then there exists a monotone decreasing function $g : I \rightarrow I$, with $g(e) = e$, such that*

$$F_2(x, y) = \begin{cases} \min(x, y), & \text{if } y < g(x) \\ \max(x, y), & \text{if } y > g(x) \\ \min(x, y) \text{ or } \max(x, y), & \text{if } y = g(x). \end{cases}$$

The following theorem gives the full characterisation of idempotent, monotone increasing, (2-ary) semigroups with neutral elements. First this was proved by Martin, Mayor and Torrens [7]. Their theorem contained a small error in the description, but essentially it was correct. In the original paper [7] there was given the following condition for g , instead of the symmetry of Γ_g . The function $g : [0, 1] \rightarrow [0, 1]$ satisfies

$$\inf\{y | g(y) = g(x)\} \leq g^2(x) \leq \sup\{y | g(y) = g(x)\} \text{ for all } x \in [0, 1]. \quad (2)$$

The authors of [8] proved that Theorem 2.2 holds if F_2 is commutative also and shown that condition (2) is not equivalent to the symmetry of Γ_g . Recently, Theorem 2.2 was reproved in an alternative way in [5] for any subinterval of \mathbb{R} .

From now on, we denote $(g \circ g)(x)$ by $g^2(x)$.

Theorem 2.2. *Let $I \subseteq \mathbb{R}$ be an arbitrary, nonempty interval. Let $F_2 : I^2 \rightarrow I$ be associative, monotone increasing, idempotent and has a neutral element $e \in I$*

if and only if there exists a decreasing function $g : I \rightarrow I$ with $g(e) = e$ such that the completed graph Γ_g is symmetric and

$$F_2(x, y) = \begin{cases} \min(x, y), & \text{if } y < g(x) \text{ or } y = g(x) \text{ and } x < g^2(x) \\ \max(x, y), & \text{if } y > g(x) \text{ or } y = g(x) \text{ and } x > g^2(x) \\ \min(x, y) \text{ or } \max(x, y), & \text{if } y = g(x) \text{ and } x = g^2(x). \end{cases} \quad (3)$$

Moreover, $F_2(x, y) = F_2(y, x)$ except perhaps the set of points (x, y) satisfying $y = g(x)$ and $x = g^2(x) = g(y)$.

For every (X, F_n) is an n -semigroup having neutral element e , then one can assign a semigroup by $F_2(a, b) = F_n(a, e, \dots, e, b)$ for every $a, b \in X$. This operation will be denoted by \mathcal{F} . Our main theoretic result in Section 3 is the following:

Theorem 2.3. *For an ordered set X the operation \mathcal{F} creates bijection between the set of idempotent, monotone, associative functions on X having neutral elements and the set of n -associative, idempotent, monotone functions on X having neutral elements.*

We will get the following result as an easy consequence of our investigation.

Theorem 2.4. *Let I be a nonempty interval. For $n \geq 2$ let $F_n : I^n \rightarrow I$ be n -associative, monotone increasing, idempotent n -ary semigroup which has a neutral element $e \in I$. Then F_n is quasitrivial.*

Applying Theorems 2.3 and 2.2 we can obtain a practical method to calculate the value of $F_n(a_1, \dots, a_n)$ for any $a_1, \dots, a_n \in I$, where $I \subset \mathbb{R}$ is an interval.

For every decreasing function $g : I \rightarrow I$ a pair (a, b) is called *critical* if $g(a) = b$ and $g(b) = a$. By Theorem 2.2 and Lemma 3.7, for every idempotent, monotone semigroup (X, F_2) with neutral element there exists a unique decreasing function g satisfying (3). Theorem 2.2 shows also that F_2 commutes in every non-critical pair $(x, y) \in I^2$ (i.e. $F_2(x, y) = F_2(y, x)$). Since for a critical pair (a, b) the value of $F_2(a, b)$ and $F_2(b, a)$ can be independently chosen from g we have two cases. We might have that F_2 commutes in a, b or not. A pair (a, b) is called *extra-critical* if critical and $F_2(a, b) \neq F_2(b, a)$. We note that being critical or extra-critical are both symmetric relations.

Finally, in order to simplify notation and give a compact way to express a value of F_n we introduce the following. The set of entries $\{a_1, \dots, a_n\}$ of F_n is denoted by A . The smallest and the largest element of A is denoted by c and d , respectively. Further there exists $1 \leq i \leq j \leq n$ such that $a_i = c$ or d , $a_j = c$ or d and $a_k \neq c$ and d for every $1 \leq k < i$ and $j < k \leq n$. We write $e_1 = a_i$ and $e_2 = a_j$.

It was proved in [3] that for any totally ordered set X if (F_n, X) is an n -ary semigroup with a neutral element e , then F_n is derived from a binary semigroup denoted by F_2 , where

$$F_2(a, b) = F_n(a, e, \dots, e, b). \quad (4)$$

Theorem 2.5. *Let $F_n : I^n \rightarrow I$ be an n -associative, idempotent function with neutral element. Assume that F_n is monotone in its first and last coordinates. If (c, d) is not an extra-critical pair, then $F_n(a_1, \dots, a_n) = F_2(c, d)$. If (c, d) is an extra-critical pair, then $F_n(a_1, \dots, a_n) = F_2(e_1, e_2)$.*

Now we point out three important consequences of Theorem 2.5. First we generalise Czogala-Drewniak's Theorem (Theorem 2.1) as follows.

Theorem 2.6. *Let I be an arbitrary nonempty real interval. If a function $F_n : I^n \rightarrow I$ is n -associative, idempotent, monotone which has a neutral element $e \in I$, then there exists a monotone decreasing function $g : I \rightarrow I$ with $g(e) = e$ such that Γ_g is symmetric and*

$$F_n(a_1, \dots, a_n) = \begin{cases} c, & \text{if } c < g(d) \\ d, & \text{if } c > g(d) \\ c \text{ or } d, & \text{if } c = g(d), \end{cases}$$

where c and d denotes the minimum and the maximum of set $A = \{a_1, \dots, a_n\} \subset \mathbb{R}$, respectively.

We note that the generalization of Theorem 2.2 is essentially stated in Theorem 2.5. In [8] the authors investigated idempotent uninorms, which are associative, commutative, monotone functions with a neutral element and idempotent also. We introduce n -uninorms, which are n -associative, commutative, monotone functions with neutral element. Here we show the generalization of Theorem 3. [8] for n -ary operations.

Theorem 2.7. *An n -ary operator U_n is an idempotent n -uninorm on $[0, 1]$ with neutral element $e \in [0, 1]$ if and only if there exists a decreasing function $g : [0, 1] \rightarrow [0, 1]$ with fixed point e and with symmetric graph Γ_g such that*

$$U_n(a_1, \dots, a_n) = \begin{cases} c & \text{if } c < g(d) \text{ or } d < g(c) \\ d & \text{if } c > g(d) \text{ or } d > g(c) \\ c \text{ or } d & \text{if } c = g(d) \text{ and } d = g(c), \end{cases} \quad (5)$$

where c and d as in Theorem 2.6. Moreover, if (c, d) is a critical pair ($c = g(d), d = g(c)$), then the value of $U_n(a_1, \dots, a_n)$ can be chosen c or d arbitrarily and independently from other critical pairs.

We may generalise our concept in the following way. Let us define

$$X^* = \bigcup_{n \in \mathbb{N}} X^n$$

the space of finite length words of alphabet X . Multivariate function $F : X^* \rightarrow X$ is *associative* if it satisfies

$$F(\mathbf{x}, \mathbf{x}') = F(F(\mathbf{x}), F(\mathbf{x}'))$$

for all $\mathbf{x}, \mathbf{x}' \in X^*$. It is easy to check that $F|_{X^n}$ is n -associative for every $n \in \mathbb{N}$. Idempotency, monotonicity and the neutral element properties of F can be defined as they hold for every $n \in \mathbb{N}$.

Theorem 2.8. *Let I be a nonempty real interval. Then $F : I^* \rightarrow I$ is associative, idempotent, monotone and has a neutral element if and only if there is a decreasing function $g : I \rightarrow I$ with symmetric completed graph Γ_g such that $F|_{X^2}$ satisfies (3). Furthermore F must be monotone increasing in each variable.*

Concerning to associativity of multivariate functions the interested reader is referred to [4], [6].

3 From n -ary to binary semigroups

We have already seen that if there is given a semigroup (X, F_2) one can easily construct an n -ary semigroup by (X, F_n) , where $F_n = \underbrace{F_2 \circ \dots \circ F_2}_{n-1}$ with the

corresponding property as F_2 (see Lemma 3.1). The following lemma is an easy consequence of the definitions.

Lemma 3.1. *Let X be a totally ordered set and (X, F_2) be semigroup. If F_2 has any of the following properties*

- (i) *monotone*
- (ii) *idempotent*
- (iii) *has a neutral element*

then the n -associative F_n has also.

We remind that in our case X is a totally ordered set. In this section we prove Theorem 2.3. Therefore the main purpose of this section is to derive properties from the n -ary semigroup to the corresponding binary semigroup. By Definition (4) of F_2 , the element e is also a neutral element of F_2 since $F_2(e, a) = F_n(e, \dots, e, a) = a = F_n(a, e, \dots, e) = F_2(a, e)$ for every $a \in X$. Using F_2 one can define a k -associative function F_k for every $k \geq 2$ by $F_k = \underbrace{F_2 \circ F_2 \circ \dots \circ F_2}_{k-1}$.

Lemma 3.2. *Let $F_n : X^n \rightarrow X$ be n -associative, idempotent and monotone in the first and the last coordinates, which is derived from a 2-associative function F_2 . Then F_2 is monotone.*

Proof. We show that if F_n is monotone in its the last coordinate then F_2 is also monotone in the last one. Let $a \in X$ arbitrary and let $b = F_{n-1}(a, \dots, a)$. In this case $F_2(b, a) = F_n(a, \dots, a) = a$. Substituting $a = F_2(b, a)$ we obtain $a = F_2(b, F_2(b, a))$. Using the same step $n - 2$ times we get that a can be

expressed as $F_{n-1}(c_1, \dots, c_{n-1})$ for some $c_1, \dots, c_{n-1} \in X$. Then $F_2(a, x) = F_n(c_1, \dots, c_{n-1}, x)$ is clearly monotone in its last coordinate.

Similar argument shows that F_2 is monotone in its first coordinate if F_n is. \square

Remark 3.3. If F_n is n -associative, idempotent and monotone in the first and the last variables, then, by Lemma 3.2, F_2 is also monotone. It is easy to show that $F_k = \underbrace{F_2 \circ \dots \circ F_2}_{k-1}$ is k -associative and monotone in each variable.

Especially, F_n is monotone in each of its variables.

Lemma 3.4. Let $F_n : X^n \rightarrow X$ be an n -associative function ($n \geq 2$). Assume F_n is idempotent and monotone in each variable and F_n has a neutral element or is derived from an associative function F_2 . Then F_2 is idempotent as well.

Proof. We prove that $F_k = F_2 \circ \dots \circ F_2$ is idempotent for every $2 \leq k \leq n$. We use backward induction. Let us assume indirectly that there exists a $3 \leq k \leq n$ such that for a given $a \in X$

$$F_{k-1}(a, \dots, a) = b \neq a$$

and by the inductive hypothesis

$$F_k(x, \dots, x) = x$$

for every $x \in X$. We note that the second condition holds for $k = n$, since F_n is idempotent. We compare the following terms:

Table 1:

$F_k(a, \dots, a, b)$	$F_k(a, \dots, a, b, b)$	$F_k(a, \dots, a, b, b, b)$	\dots	$F_k(a, b, \dots, b, b)$
$F_k(a, \dots, a, a)$	$F_k(a, \dots, a, b, a)$	$F_k(a, \dots, a, b, b, a)$	\dots	$F_k(a, b, \dots, b, a)$

The function F_k is monotone in each variable by Remark 3.3. Observe that in the table above the elements in each column only differ in the last coordinate. Hence all of the elements in the lower row are either not less or not greater than the corresponding elements found above by the monotonicity of F_k .

Now we calculate expressions in Table 1: It is clear that $F_k(a, \dots, a) = a$ by the inductive assumption. Before we continue, we present a useful lemma.

Lemma 3.5. Let a and b as above. Further let $x_1 = \dots = x_l = a$ and $x_{l+1} = \dots = x_k = b$. Then for every $\pi \in \text{Sym}(k)$ we have

$$F_k(x_1, \dots, x_k) = F_k(x_{\pi(1)}, \dots, x_{\pi(k)}).$$

Proof. Substituting $b = F_{k-1}(a, \dots, a)$ in the expression above, it is easy to see that we may rearrange a 's and b 's arbitrarily. \square

Lemma 3.6. *Let l and m be fixed and $l + m = k$. If $1 \leq m \leq k - 2$, then*

$$F_k(\underbrace{a, \dots, a}_l, \underbrace{b, \dots, b}_m) = F_l(\underbrace{a, \dots, a}_l).$$

Especially, if $m = k - 1$, then

$$F_k(a, \underbrace{b, \dots, b}_{k-1}) = a.$$

Proof. Direct calculation shows that the statement holds. Indeed

$$F_k(\underbrace{a, \dots, a}_l, \underbrace{b, \dots, b}_m) = F_k(\underbrace{a, \dots, a}_l, \underbrace{F_{k-1}(a, \dots, a), \dots, F_{k-1}(a, \dots, a)}_m).$$

Now using the associativity of F_2 and the idempotency of F_k , we obtain that $F_2(a, F_{k-1}(a, \dots, a)) = F_k(a, \dots, a) = a$. Applying Lemma 3.5 and the previous observation m times we obtain

$$F_k(\underbrace{a, \dots, a}_l, \underbrace{F_{k-1}(a, \dots, a), \dots, F_{k-1}(a, \dots, a)}_m) = F_l(a, \dots, a).$$

□

Using Lemma 3.5 and Lemma 3.6 we get that

$$\begin{aligned} F_k(a, \dots, a) &= a \\ F_k(a, \dots, a, a, b, a) &= F_k(a, \dots, a, a, a, b) = F_{k-1}(a, \dots, a), \\ F_k(a, \dots, a, b, b, a) &= F_k(a, \dots, a, a, b, b) = F_{k-2}(a, \dots, a), \\ &\vdots \\ F_k(a, b, b, \dots, b, a) &= F_k(a, a, b, \dots, b, b) = F_2(a, a), \\ F_k(a, b, \dots, b, b) &= a. \end{aligned}$$

Since in each column of the table we just change the last coordinate, we can use monotonicity. Substituting these results to the table we get that one of the following holds:

1.

$$\begin{array}{c|c|c|c|c} F_{k-1}(a, \dots, a) & F_{k-2}(a, \dots, a) & F_{k-3}(a, \dots, a) & \dots & a \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ a & F_{k-1}(a, \dots, a) & F_{k-2}(a, \dots, a) & \dots & F_2(a, a) \end{array}$$

2.

$$\begin{array}{c|c|c|c|c} F_{k-1}(a, \dots, a) & F_{k-2}(a, \dots, a) & F_{k-3}(a, \dots, a) & \dots & a \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a & F_{k-1}(a, \dots, a) & F_{k-2}(a, \dots, a) & \dots & F_2(a, a) \end{array}$$

Here the notation \uparrow (notation \downarrow) means that an element in the lower row is greater than or equal (less than or equal) to the corresponding element in the upper row. Thus

$$a \geq F_{k-1}(a, \dots, a) \geq F_{k-2}(a, \dots, a) \geq \dots \geq F_2(a, a) \geq a,$$

or

$$a \leq F_{k-1}(a, \dots, a) \leq F_{k-2}(a, \dots, a) \leq \dots \leq F_2(a, a) \leq a.$$

Both of the cases gives

$$a = F_{k-1}(a, \dots, a) = F_{k-2}(a, \dots, a) = \dots = F_2(a, a).$$

This is a contradiction, since $F_{k-1}(a, \dots, a) = b \neq a$ by our assumption. This shows that F_k is idempotent for every $2 \leq k \leq n$. \square

The following lemma provides extra information about monotone semigroups.

Lemma 3.7. *Let X be a totally ordered set. If $F_2 : X^2 \rightarrow X$ is associative, idempotent and monotone in each variable, then F_2 is monotone increasing in each variable.*

Proof. Indirectly, we assume that F_2 is not monotone increasing in each variable. Since F_2 is idempotent, F_2 cannot be monotone decreasing in both variables except if it is constant in which case it is monotone increasing. Let us assume that F_2 is increasing in the first and decreasing and not constant in the second variable. Thus there exists $x \neq y \in X$ such that $F_2(x, x) \neq F_2(x, y)$. Otherwise F_2 is constant in the second variable which is monotone increasing, contradicting our assumption.

If $x < y \in X$ and $F_2(x, y) < F_2(x, x)$, then using monotonicity, we obtain

$$\begin{aligned} x &= F_2(x, x) = F_2(x, F_2(x, x)) \leq F_2(x, F_2(x, y)) = \\ &= F_2(F_2(x, x), y) = F_2(x, y) < F_2(x, x) = x. \end{aligned}$$

The first, second, fifth and the last equalities use the idempotency of F_2 and the third one uses the associative property. This equation contradicts $F_2(x, y) < F_2(x, x)$.

If $x > y \in X$ and $F_2(x, y) > F_2(x, x)$, then similarly we obtain

$$\begin{aligned} x &= F_2(x, x) = F_2(x, F_2(x, x)) \geq F_2(x, F_2(x, y)) = \\ &= F_2(F_2(x, x), y) = F_2(x, y) > F_2(x, x) = x, \end{aligned}$$

which is a contradiction again.

One can get the same type of contradiction if we switch the role of the coordinates. Thus F_2 is monotone increasing in both variables. \square

Remark 3.8. Now we show that there are examples when we omit any of the conditions of Lemma 3.7. In these cases the statement of the lemma does not remain true.

1. Let $F_2(x, x) = x$ for $x \in \mathbb{R}$ and $F_2(x, y) = 0$ if $x, y \in \mathbb{R}, x \neq y$. Then F_2 is associative and idempotent, but not monotone in each variable.
2. Let $F_2(x, y) = 2x - y$ for $x, y \in \mathbb{R}$. Then F_2 is idempotent and monotone in each variable, but not associative and clearly not monotone increasing.
3. Let $F_2(x, y) = -x$, if $x, y > 0$, and $F_2(x, y) = 0$ otherwise. Then F_2 is associative, since $F_2(x, F_2(y, z)) = F_2(F_2(x, y), z) = 0$ and F_2 is monotone decreasing in each variable but F_2 is not idempotent.

Corollary 3.9. *If $F_n = \underbrace{F_2 \circ F_2 \circ \dots \circ F_2}_{n-1}$ is n -associative, idempotent and monotone in the first and in the last variables, then F_n is monotone increasing in each variable.*

Moreover, $F_k = \underbrace{F_2 \circ F_2 \circ \dots \circ F_2}_{k-1}$ is monotone increasing for every $2 \leq k \leq n$.

Proof. By definition, F_2 is associative. Since F_n is monotone in each variable, by Lemma 3.2, F_2 is monotone in each variable. By Lemma 3.4, F_2 is idempotent. Thus by Lemma 3.7, it is monotone increasing. Thus $F_k = F_2 \circ F_2 \circ \dots \circ F_2$ is also monotone increasing for every $2 \leq k \leq n$. \square

If F_n is n -associative and has a neutral element, then there exists F_2 such that $F_n = F_2 \circ F_2 \circ \dots \circ F_2$. Using the results of this section we prove the following proposition.

Proposition 3.10. *Let (X, F_n) be an n -ary semigroup, which is monotone, idempotent and has a neutral element. Then F_n is derived from a binary semigroup (X, F_2) , where F_2 is also monotone idempotent and it also has a neutral element. Moreover F_n is monotone increasing in each variables.*

Proof. Since F_n is idempotent, n -associative and has a neutral element, it follows from [3] that $F_n = F_2 \circ \dots \circ F_2$, where $F_2 : X^2 \rightarrow X$ is associative. By Lemma 3.2, Lemma 3.4 and Lemma 3.7 F_2 is monotone increasing and idempotent. Before Lemma 3.2 we mentioned that in this case F_2 has a neutral element, as well. \square

Proof of Theorem 2.3. By Proposition 3.10, if there is n -associative F_n which is monotone, idempotent and has a neutral element, then F_n derived from an associative F_2 determined by $F_2(a, b) = F_n(a, e, \dots, e, b)$ which is monotone, idempotent and has a neutral element. Since F_n is idempotent, F_2 is also. Therefore

$$F_n(a, \dots, a, b) = F_n(a, e, \dots, e, b) = F_n(a, b, \dots, b) = F_2(a, b) \quad (6)$$

We remind that this operation which assigns an F_2 for every F_n as above was denoted by \mathcal{F} .

First we show that \mathcal{F} is surjective. By Lemma 3.1 and Proposition 3.10 for every F_2 there exists an F_n satisfying (6).

The operation \mathcal{F} is injective since if $F_2(a, b) \neq F'_2(a, b)$ for some $a, b \in X$, then $F_n(a, \dots, a, b) \neq F'_n(a, \dots, a, b)$ by (6). This finishes the proof of Theorem 2.3. \square

Remark 3.11. Using Corollary 3.9 we may weaken the assumptions of Theorem 2.3 for F_n which is monotone in the first and the last variables instead of F_n monotone in each variable since, by Proposition 3.10, these are *equivalent*.

Lemma 3.12. *Let (X, F_n) be a totally ordered n -ary semigroup derived from (X, F_2) , where F_2 is idempotent, associative, monotone increasing and have a neutral element on X . Then*

$$\begin{aligned} F_n(a, y_1, \dots, y_{n-2}, b) &= F_2(a, b) \\ F_n(b, y_1, \dots, y_{n-2}, a) &= F_2(a, b) \end{aligned} \tag{7}$$

for every $a \leq y_1, \dots, y_{n-2} \leq b$.

Proof. By Theorem 2.3, F_n is monotone and the statement directly comes from (6).

4 Proof for the main results

Proof of Theorem 2.4: It follows from Proposition 3.10 that F_n is derived from an associative function F_2 . Moreover F_2 is monotone, idempotent and has a neutral element. Therefore we may apply Theorem 2.2 in a special form we obtain that F_2 is a choice function (i.e: (X, F_2) is quasitrivial). Since F_n is obtained as the composition of $n-1$ copies of F_2 we get that F_n is also a choice function. \square

Proof of Theorem 2.5. Let us assume first that c and d commute with every element of A . In this case we may assume using the idempotency of F_2 that $F_n(a_1, \dots, a_n) = F_n(c, a'_2, \dots, a'_{k-1}, d)$ and $c \leq a'_i \leq d$ for every $2 \leq i \leq k-1$. By Proposition 3.10 we can apply Lemma 3.12 and we get $F_n(a_1, \dots, a_n) = F_2(c, d)$.

Let us assume that d does not commute with an element of A but c commutes with all of them. In this case $g(d) \in A$ is the one not commuting with d . Since c is the smallest element of A we get $c < g(d)$. Further, d is the largest element of A and g is decreasing so $g(a_i) > c$ for $1 \leq i \leq n$. Theorem 2.2 gives us that $F_2(c, a_i) = F_2(a_i, c) = c$ for $1 \leq i \leq n$. Therefore $F_n(a_1, \dots, a_n) = c$. Since $F_2(c, a_i) = c$ for every i we get $F_n(a_1, \dots, a_n) = F_2(c, d) = c$. Similar argument shows that $F_n(a_1, \dots, a_n) = d = F_2(c, d)$ if we switch the role of c and d .

Finally, let us assume that neither c nor d commutes with every element of A . In this case the set A contains $g(c)$ and $g(d)$ and $g(g(c)) = c$, $g(g(d)) = d$. We claim that $g(c) = d$ and $g(d) = c$. Indeed, if $g(c) \in A$, then $g(c) \leq d$ since d is the largest element of A , and similarly $g(d) \geq c$. Since g is monotone and $g(g(d)) = d$, we get $d = g(g(d)) \leq g(c)$. Therefore $g(c) = d$. Similarly using $c = g(g(c)) \geq g(d)$ we get $g(d) = c$. What we obtained is that (c, d) is an extra-critical pair in this case.

Now c and d are the elements in A that do not commute which implies that c and d commutes with all other element of A .

By definition of e_1 and e_2 (see Section 2), e_1 and e_2 are the value of the first and the last appearance of c or d , respectively. Since e_1 commutes with its left neighbours and e_2 commutes with its right, we may assume that $a_1 = e_1$ and $a_n = e_2$. We get the following cases:

- (i) If $e_1 \neq e_2$ then by Lemma 3.12

$$F_n(e_1, \dots, e_2) = F_2(e_1, e_2).$$

- (ii) If $e_1 = e_2$ then we show that

$$F_n(e_1, \dots, e_2) = F_2(e_1, e_2) = e_1.$$

Using Lemma 3.12 for arbitrary number of variables we get that every subterm of a_1, \dots, a_n consisting of elements lying strictly between c and d can be eliminated. Thus, one can write $F_n(a_1, \dots, a_n) = F_k(b_1, \dots, b_k)$ where $k \leq n$ and $b_i = c$ or d for every $1 \leq i \leq k$ and, in our case, $b_1 = b_k = e_1$. Since F_2 is idempotent we may assume $b_i \neq b_{i+1}$ for $1 \leq i \leq k-1$. Using the idempotency and the associativity of F_2 again we have $F_2(F_2(c, d), F_2(c, d)) = F_2(c, d)$ and $F_2(F_2(d, c), F_2(d, c)) = F_2(d, c)$. Therefore $F_k(b_1, \dots, b_k)$ can be reduced to one of the form $F_3(c, d, c), F_3(d, c, d)$.

If $F_2(c, d) = c$, then $F_3(c, d, c) = F_2(F_2(c, d), c) = F_2(c, c) = c = e_1$.

If $F_2(c, d) = d$, then $F_3(c, d, c) = F_2(F_2(c, d), c) = F_2(d, c)$. Since c and d do not commute we get $F_2(d, c) = c = e_1$.

Similarly, one can verify that $F_3(d, c, d) = d = e_1$. This finishes the proof of Theorem 2.5. \square

5 Concluding remarks

In this paper we investigated the n -ary associative, idempotent, monotone functions $F_n : X^n \rightarrow X$ which has a neutral element. We showed that such an F_n in general setting when the underlying set X is totally ordered implies the existence of binary functions $F_2 : X^2 \rightarrow X$ with similar properties such that F_n

is derived by F_2 . We summarise the results of section 3 in the following table.

Properties of F_n		Properties of F_2
n – assoc. with a neutral element	\implies	assoc. with a neutral element
Now we assume $F_n = F_2 \circ \dots \circ F_2$:		
n – assoc., idempotent, monotone	\implies	monotone
n – assoc., idempotent, monotone	\implies	idempotent
n – assoc., idempotent, monotone	\implies	monotone increasing
Some easy observations shows:		
n – associative	\Leftarrow	associative
monotone increasing	\Leftarrow	monotone increasing
idempotent	\Leftarrow	idempotent
has a neutral element	\Leftarrow	has a neutral element
Thus:		
n – assoc., idempotent, mon. incr.	\iff	assoc., idemp., mon. incr.

In the main results we obtain a characterization of n -associative, idempotent, monotone functions on any (not necessarily bounded) subinterval of \mathbb{R} in the spirit of the characterization of the binary case. We also generalise the classical Czogala-Drewniak's theorem. In addition, we get that every n -associative, idempotent, monotone function with a neutral element must be quasitrivial.

Further improvement would be based on the elimination of any of the properties of F_n . The most crucial property seems to be that F_n has a neutral element since all of our results based on this condition otherwise F_n is not necessarily derived from F_2 . On the other hand, in [7] one can be found a characterization of associative, monotone increasing, idempotent binary functions without assumption on the neutral element. Therefore we offer the following question for further investigation.

Question 5.1. *How can we characterise the n -associative, monotone, idempotent function F_n on a subinterval of \mathbb{R} ?*

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